Module1

Welcome to the first module of the course. It is indeed an exciting event to share with you the subject that has lot to offer both from theoretical side and practical aspects.

To begin with, we will keep the things simple and straight to excite you about the course. You can think of this module as a warm up game before the big match starts.

We do not assume you to have a preliminary background of linear algebra. Therefore, we will build the basic blocks from scratch. So, lets start.

Lecture-1

We Shall begin with some simple examples.
Example1.1 Consider a calculator company which produces a scientific calculator and a graphing calculator. Long-term projections indicate an expected demand of at least 1000 scientific and 800 graphing calculators each month. Because of limitations on production capacity, no more than 2000 scientific and 1700 graphing calculators can be made monthly. To satisfy a supplying contract, a total of atleast 2000 calculators must be supplied each month. If each scientific calculator sold results in Rs.120 profit and each graphing calculator sold produces a Rs.150 profit, how many of each type of calculators should be made monthly to maximize the net profit.

Let's make a mathematical model for the problem. Suppose, x is the number of scientific calculators produced and y is the number of graphing calculators produced.
Since the company can not produce negative calculators; it will produce either no or positive numbers of calculators only, we must have $x \geq 0$, $y \geq 0$. But, keeping in mind the demand of each type of calculator we have

$$x \geq 1000$$

$$y \geq 800.$$

These two submused $x \geq 0$, $y \geq 0$. Thus, we can drop a superficial requirement of $x \geq 0$, $y \geq 0$ from our modeling. But we caution you that this may not be the case always and many a times it makes sense to assume variables be non-negative.
Another issue is to take care of production capacity of the calculators. So,

$$x \leq 2000$$

$$y \leq 1700.$$

Looking at the supply restriction, we must have

$$x + y \geq 2000.$$

There is no other information which is in the form of restrictions/conditions in the question. Then, we move on to look at the profit part. The net profit from sale of x scientific and y graphing calculators is

$$120x + 150y$$

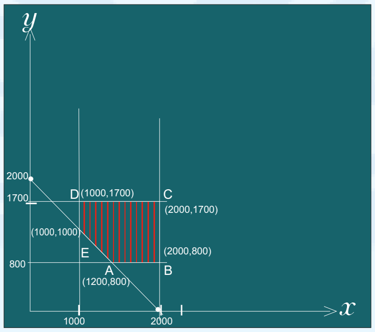
Finally, we are able to identify our aim (or objective) so as to maximize the net profit per month, which means,

$$\max\ 120x + 150y$$

but our x and y must satisfy the aforementioned restrictions. In conclusion, the ultimate model of the problem becomes

$$\begin{aligned} &\max\ 120x + 150y \\ &\text{subject to} \\ &\qquad 1000 \leq x \ \leq\ 2000 \\ &\qquad\ \ 800 \leq y \ \leq\ 1700 \\ &\qquad\quad\ x + y \ \geq\ 2000. \end{aligned}$$

Since the model is set up in two dimensional plane, we can actually see what we are intending to do. Let us draw the region formed by the restrictions on x and y.



The portion shaded in red is the collection of points (x, y) satisfying

$$\begin{aligned} 1000 \leq x \ &\leq\ 2000 \\ 800 \leq y \ &\leq\ 1700 \\ x + y \ &\geq\ 2000 \end{aligned}$$

Now, what we want is to maximize $120x + 150y$ in this domain(i.e. red shaded region) only. In other words we wish to find a pair (x^*, y^*) which lies in the shaded region and provide the maximum value of our goal function $120x + 150y$.

Though there are infinite points in the identified region and testing of each and every one of them is not a good idea, we need to design a strategy which can find answer to our objective in a more simple way. This is what we shall be intending to do in the next few modules to follow but for the time being let us see how the objective function behaves on the 'corner points' of the shaded region.

The corner points are A, B, C, D, E and their co-ordinates are described in graph. Let us tabulate the results

Point	Value of 120x+150y
A:(1200,800)	264,000
B:(2000,800)	360,000
C:(2000,1700)	510,000
D:(1000,1700)	375,000
E:(1000,1000)	270,000

From these five values(only), we can clearly make out that the best one is $C(2000, 1700)$ with the maximum value of $120x + 150y$ as 510,000.

Although, we have not checked it but just by looking at the above graph we can atleast guess and get confident about our guess that no matter what other point (x, y) we select in the shaded region, the maximum value still remains 510,000 and no further improvement(or enhancement) is possible in it within the shaded region.

We can conclude that the company should manufacture 200 scientific calculators and 1700 graphing calculators to get the best monthly profit of Rs 510,000 within the limitations of the company.
Let us see few more examples.

Example1.2 Suppose you need to buy some cabinets for a room. There are two types of cabinets that you have liked in the market, say X and Y. Each unit of cabinet X costs you Rs 15000 and requires 6 square feet of floor space, and holds 8 cubic feet of files. On the other hand each unit of cabinet Y Costs Rs 12000, requires 8 square feet of floor space and holds 12 cubic feet of files. You have been given Rs 140,000 for this purchase, though you may not spend all. The office has room for no more than 72 square feet of cabinets. How many of each model you must buy in order to maximize the storage volume?

As done in previous example, let us introduce two variables x and y for numbers of models X and Y cabinets purchased respectively. Naturally, $x \geq 0$, $y \geq 0$.
If we look at the cost restriction then, we must have,

$$15000x + 12000y \leq 140000$$

or equivalently

$$15x + 12y \leq 140.$$

Now, think of floor space restriction. Then,

$$6x + 8y \leq 72 \text{ or } 3x + 4y \leq 36$$

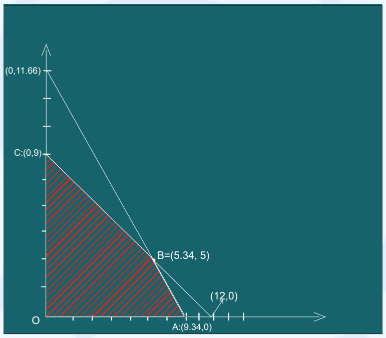
There are no more restrictions to take care off. Now see what is our objective. It is to maximize the requisite volume. By volume data of X and Y, we have

$$\text{Volume} = 8x + 12y$$

Thus, we can easily say that the model that we need to solve is the following

$$\begin{array}{rcl} \max & 8x + 12y & \\ \text{subject to} & & \\ 15x + 12y & \leq & 140 \\ 3x + 4y & \leq & 36 \\ x, y & \geq & 0. \end{array}$$

We try and plot the graph of restrictions on x and y.



The shaded region shows all points (x, y) which satisfy the conditions

$$\begin{array}{rcl} 15x + 12y & \leq & 140 \\ 3x + 4y & \leq & 36 \\ x, y & \geq & 0. \end{array}$$

The point B is the point of intersection of lines $150x + 12y = 140$ and $3x + 4y = 36$.

Again what we see is that there are simply too many points in shaded region to get the final best answer. It is again unreasonable to check each and every one of them individually. But we can see what value our objective function $8x+12y$ takes on the four corner points.

Point	Value of 8x+12y
O:(0,0)	0
A:(9.34,0)	74.72
B:(5.34,5)	102.72
C:(0,9)	108

The maximum value among these four is 108 at the point $C : (0, 9)$. Unlike previous example it is little difficult to get the idea that among all points in the shaded region, C

will still yield the best objective value. But believe us (at least for the time being) that it will indeed be so.
Hence you can buy 9 cabinets of type Y for the office to maximize the volume capacity.

There are few things that we would like you to observe from above two examples (although, we agree that 'two' examples is not a large number for any conclusion).

(i) The shaded region is non-empty and closed bounded polyhedron.

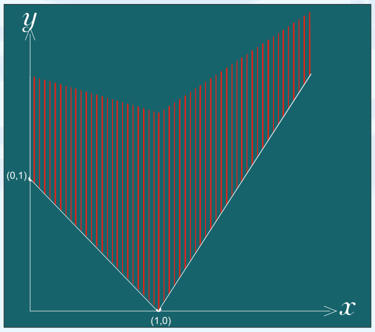
(ii) The best value for objective function is at the corner of the shaded region.
Is it a fluke or phenomenas observed sometimes or always happening events or simply we do not know!

Let us see the following model.
Now, we have deliberately avoided the language of the problem but concentrated on its model. Perhaps, I am confident that you all can easily build the model for a given situation!!

Example1.3 max $7x + 5y$
subject to

$$\begin{aligned} x + y &\geq 1 \\ 2x - y &\leq 2 \\ x, y &\geq 0. \end{aligned}$$

The graphical description is as follows:



Now, observe that the shaded region is closed non-empty but not bounded from above. Is there anything wrong with the model! No, nothing; rest assure that it can happen that the domain region where you have to find an answer of your problem may exhibit unbounded behaviour. But one thing is still here, the shaded region is a polyhedron.

Now, look at the objective function $7x + 5y$. By increasing the value of y upwards, in the region only, we can increase the value of the objective function to infinity. Thus, we do not have the best answer or point in a classical sense. In fact the objective-value is unbounded within the shaded region.

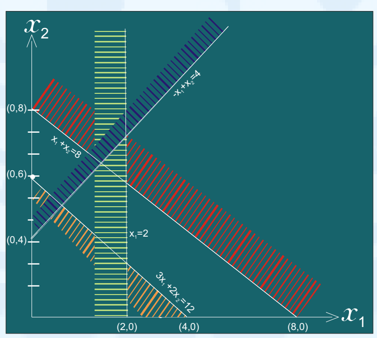
Now, suppose instead we take an objective function as $7x - 5y$ (just a change of sign with y-coefficient). Then the point (1,0) yields the maximum value for this function. Think how do we claim it! We will definitely answer this query but after a while.

What we observe is that, if the domain region of search for the 'best' point is unbounded then, the objective value can both be unbounded or finite. Let us look at some other models to exhibit more cases.

Example1.4 Consider

$$\max x_1 + \tfrac{1}{2}x_2$$
subject to
$$\left.\begin{array}{rcl} 3x_1 + 2x_2 & \leq & 12 \\ -x_1 + x_2 & \geq & 4 \\ x_1 & \leq & 2 \\ x_1 + x_2 & \geq & 8 \\ x_1,\ x_2 & \geq & 0 \end{array}\right\} \qquad (1)$$

Repeating the procedure highlighted in previous examples, we first determine the set of points (x_1, x_2) satisfying (1).



The **red** depicts points satisfying $x_1 + x_2 \geq 8$, while the **orange** shaded region is those points satisfying $3x_1 + 2x_2 \leq 12$. The **green** for $x_1 \leq 2$ and **blue** for $-x_1 + x_2 \geq 4$.

What we observe is that there is no region which is the intersection of all four colors. In other words, we mean to say that there is no point (x_1, x_2) satisfying all four inequalities and non-negativity restriction in (1) simultaneously.

In such a case, we call the problem as infeasible. The terminology 'infeasible' comes because the points satisfying the restrictions on search domain (the shaded region) are called feasible points and the domain (or shaded region) is called feasible set.

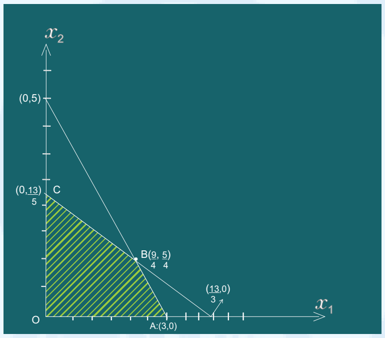
Thus if feasible set in a problem is an empty set (like in above example), it is natural to call the problem 'infeasible'.

Moving ahead, let us consider one more example before we summarize all this discussion.

Example1.5 Consider

$$\begin{aligned}
&\max\ 6x_1 + 10x_2 \\
&\text{subject to} \\
&\qquad 3x_1 + 5x_2 \leq 13 \\
&\qquad 5x_1 + 3x_2 \geq 15 \\
&\qquad x_1, x_2 \geq 0.
\end{aligned}$$

The graphical depiction of the model is as follows.



Let us compute objective values at four corner points

Point	objective value
O:(0,0)	0
A:(3,0)	18
B:($\frac{9}{4}$, $\frac{5}{4}$)	26
C:(0,$\frac{13}{5}$)	26

Note that objective value is maximum for both corners B and C (which are adjacent corners). Now, suppose we choose any $\lambda \in (0,1)$ and take a point.

$$\lambda\left(\tfrac{9}{4},\tfrac{5}{4}\right) + (1-\lambda)\left(0,\tfrac{13}{5}\right) = \left(\tfrac{9}{4}\lambda, \tfrac{5}{4}\lambda + (1-\lambda)\tfrac{13}{5}\right) \tag{2}$$

Actually this is a parametric equation of the line segment BC, i.e. any point on BC is type (2) only for some $\lambda \in (0,1)$.

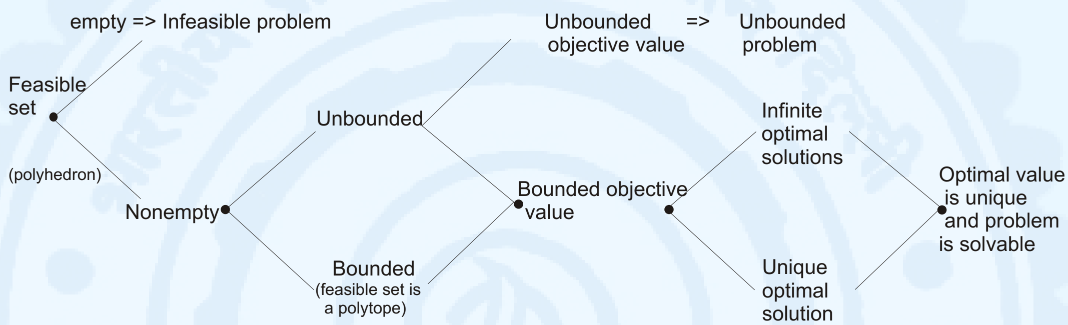
Compute the object value at the point in (2); it is equal to

$$6\left(\tfrac{9}{4}\lambda\right) + 10\left(\tfrac{5}{4}\lambda + \tfrac{13}{5}(1-\lambda)\right) = 26,\ \forall\ \lambda \in (0,1).$$

Consequently the objective value on all points on the line segment BC including B and C remains 26. And believe me it is the maximum value of objective function within the shaded region.

This is the case when we do not have a unique optimal solution but infinite optimal solutions. However, note that optimal value remains unique.

Our observation from above examples can be summarized as follows:



The other two important observations are:

- Optimal solution (if exists) is not in the interior of the feasible set(shaded region)!
- Among all optimal solutions (if exist) one of the optimal solution must be a "corner point" of the feasible set.
- A problem can have either no solution, or a unique optimal solution or an infinite optimal solution; no other case is possible.

Although we have made all these conclusions based on few simple examples in two-dimensional, but rest assure we will learn in next few lectures to prove them in n-dimensional finite real space.

Let us give a formal description of what class of problems we will concentrate on in the chapters to come in this course. We will be studying problems having following structure.

$$
\text{(P)} \qquad \begin{array}{rl} & \max z = c^T x \\ \text{subject to} & \\ Ax & \leq \; b \\ x & \geq \; 0 \end{array}
$$

where $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$.
Though we have written all inequalities in $\leq$ form but some of them could be $\geq$ or $=$ types also.

The restrictions on x restrict the domain in which we have to search a solution(if exists) for (P) is described by

$$S = \{x \in \mathbb{R}^n | Ax \leq b,\ x \geq 0\}.$$

is called the feasible set of (P). Any decision vectors $x \in S$ is called feasible solution of (P). The decision variable $x^* \in S$ such that $c^T x^* \geq C^T x$, $\forall x \in S$, is called an optimal solution of (P) and $c^T x^*$ is called the optimal value of z. The function $c^T x$ to optimize (here maximization) is called the objective function of (P) and the inequalities describing the feasible set S of (P), i.e., $Ax \leq b$, are called the constraints of (P). The restriction, $x \geq 0$, is called the non-negativity condition on x.

Since all the functions (i.e., objective function and m constraints are linear functions of decision variable x, the problems of the type (P) called **linear programming problems** or simply LPP.

Note that as it is not necessary to have all constraints in (P) to be $\leq$ type (we have used it for notational convenience), it is also not necessary to have max problem only. We can work our for min problem because

$$\min\ z = c^T x = -[max(-z) = -c^T x].$$

Thus, we may solve max problem with negative objective coefficients (than the one given), and later on if $(-c^T)(x^*)$ is the optimal value of max problem, then $c^T x^*$ is the optimal value of the original min linear programming problem.